

REMARKS ON HALL ALGEBRAS OF TRIANGULATED CATEGORIES

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ABSTRACT. By a detailed investigation of the proof in [20], we give the explicit relations among several versions of derived Hall algebras in [19], [20] and [9].

1. INTRODUCTION

Let k be a finite field of q elements and \mathcal{A} be a finitary k -category, i.e., a (small) abelian category satisfying: (1) $|\mathrm{Hom}_{\mathcal{A}}(M, N)| < \infty$; (2) $|\mathrm{Ext}_{\mathcal{A}}^1(M, N)| < \infty$ for any $M, N \in \mathcal{A}$. The Hall algebra $\mathcal{H}(\mathcal{A})$ associated to a finitary category \mathcal{A} is originally defined by Ringel in [17] in order to realize quantum groups. In the simplest version, it is an associative algebra, which, as a \mathbb{Q} -vector space, has a basis consisting of the isomorphism classes $[X]$ for $X \in \mathcal{A}$ and has the multiplication $[X] * [Y] = \sum_{[L]} g_{XY}^L [L]$, where $X, Y, L \in \mathcal{A}$ and $g_{XY}^L = |\{M \subset L \mid M \simeq X \text{ and } L/M \simeq Y\}|$. The structure constant g_{XY}^L is called *Hall number* and the algebra $\mathcal{H}(\mathcal{A})$ now is called Ringel-Hall algebra. The Ringel-Hall algebras have been developed many variants (see [4]) as a framework involving the categorification and the geometrization of Lie algebras and quantum groups in the past two decades (for example, see [17, 10, 15, 11, 13, 14]).

It is easy to verify the associativity property of the Ringel-Hall algebra from an abelian category. As a generalization, aiming at a global realization of quantum groups, two kind Hall algebras associated to derived categories or triangulated categories with some homological finiteness conditions were given by Toën [19] and by Kontsevich-Soibelman [9]. The associativity property of these algebras become a uneasy thing. In [20], we have given a direct proof for Toën's derived Hall algebras by counting some invariants which appear in the octahedral axiom. Another new development is the theory of motivic Hall algebras, see Bridgeland [1], Joyce ([3, 4, 5, 6]) and Kontsevich-Soibelman [9], in particular, the Hall algebra in [9] is not only for triangulated category but also in motivic version.

In this note, we will point out that the method of the paper [20] provides two symmetries associated with the octahedral axiom, actually they are equivalent. the first symmetry implies the associativity of the derived Hall algebra in the sense of Toën, the second symmetry implies the associativity of the Hall algebra in the sense of Kontsevich-Soibelman. Therefore we obtain an explicit isomorphism between the two Hall algebras. In the last section we use our method to give a new proof of the associativity property of the motivic Hall algebra of Kontsevich-Soibelman [9].

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2. A GENERALIZATION OF THE RIEDTMANN-PENG FORMULA FOR HOMOLOGICALLY FINITE TRIANGULATED CATEGORIES

We recall some notations and results in [20]. Let k be a finite field with q elements and \mathcal{C} a (left) homologically finite k -additive triangulated category with the translation (or shift) functor $T = [1]$ satisfying the following conditions (see [20])

- (1) the homomorphism space $\text{Hom}(X, Y)$ for any two objects X and Y in \mathcal{C} is a finite dimensional k -space;
- (2) the endomorphism ring $\text{End}X$ for any indecomposable object X in \mathcal{C} is a finite dimensional local k -algebra;
- (3) \mathcal{C} is (left) locally homological finite, i.e., $\sum_{i \geq 0} \dim_k \text{hom}(X[i], Y) < \infty$ for any X and Y in \mathcal{C} .

Note that the first two conditions imply the validity of the Krull–Schmidt theorem in \mathcal{C} , which means that any object in \mathcal{C} can be uniquely decomposed into the direct sum of finitely many indecomposable objects up to isomorphism.

For any X, Y and Z in \mathcal{C} , we will use fg to denote the composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, and $|S|$ to denote the cardinality of a finite set S .

Given $X, Y; L \in \mathcal{C}$, put

$$W(X, Y; L) = \{(f, g, h) \in \text{Hom}(X, L) \times \text{Hom}(L, Y) \times \text{Hom}(Y, X[1]) \mid \\ X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1] \text{ is a triangle}\}.$$

There is a natural action of $\text{Aut}X \times \text{Aut}Y$ on $W(X, Y; L)$. The orbit of $(f, g, h) \in W(X, Y; L)$ is denoted by

$$(f, g, h)^\wedge := \{(af, gc^{-1}, ch(a[1])^{-1}) \mid (a, c) \in \text{Aut}X \times \text{Aut}Y\}.$$

The orbit space is denoted by $V(X, Y; L) = \{(f, g, h)^\wedge \mid (f, g, h) \in W(X, Y; L)\}$. The radical of $\text{Hom}(X, Y)$ is denoted by $\text{radHom}(X, Y)$ which is the set

$$\{f \in \text{Hom}(X, Y) \mid gfh \text{ is not an isomorphism for any } g : A \rightarrow X \text{ and} \\ h : Y \rightarrow A \text{ with } A \in \mathcal{C} \text{ indecomposable}\}.$$

For any $L \xrightarrow{n} Z[1]$, there exist the decompositions $L = L_1(n) \oplus L_2(n)$, $Z[1] = Z_1[1](n) \oplus Z_2[1](n)$ and $b \in \text{Aut}L$, $d \in \text{Aut}Z$ such that $bn(d[1])^{-1} = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}$ and the induced maps $n_{11} : L_1(n) \rightarrow Z_1[1](n)$ is an isomorphism and $n_{22} : L_2(n) \rightarrow Z_2[1](n)$ belongs to $\text{radHom}(L_2(n), Z_2[1](n))$. The above decomposition only depends on the equivalence class of n up to an isomorphism. Let $\alpha = (l, m, n)^\wedge \in V(Z, L; M)$, the classes of α and n are determined to each other in $V(Z, L; M)$. We may denote n by $n(\alpha)$ and $L_1(n)$ by $L_1(\alpha)$ respectively.

Denote by $\text{Hom}(X, Y)_Z$ the subset of $\text{Hom}(X, Y)$ consisting of the morphisms whose mapping cones are isomorphic to Z . For $X, Y \in \mathcal{C}$, we set

$$\{X, Y\} := \prod_{i \geq 0} |\text{Hom}(X[i], Y)|^{(-1)^i}.$$

By checking the stable subgroups of automorphism groups, we have the following proposition.

Proposition 2.1. [20, Proposition 2.5] For any M, L and Z in \mathcal{C} , the following equalities hold

$$\frac{|\mathrm{Hom}(M, L)_{Z[1]}|}{|\mathrm{Aut} L|} \cdot \frac{\{M, L\}}{\{Z, L\} \cdot \{L, L\}} = \sum_{\alpha \in V(Z, L; M)} \frac{|\mathrm{End} L_1(\alpha)|}{|\mathrm{Aut} L_1(\alpha)|}$$

$$\frac{|\mathrm{Hom}(Z, M)_L|}{|\mathrm{Aut} Z|} \cdot \frac{\{Z, M\}}{\{Z, L\} \cdot \{Z, Z\}} = \sum_{\alpha \in V(Z, L; M)} \frac{|\mathrm{End} L_1(\alpha)|}{|\mathrm{Aut} L_1(\alpha)|}$$

Using the proposition, one can easily deduce the following corollary.

Corollary 2.2. For any X, Y and L in \mathcal{C} , we have

$$\frac{|\mathrm{Hom}(Y, X[1])_{L[1]}|}{|\mathrm{Aut} X|} \cdot \frac{\{Y, X[1]\}}{\{X, X\}} = \frac{|\mathrm{Hom}(L, Y)_{X[1]}|}{|\mathrm{Aut} L|} \cdot \frac{\{L, Y\}}{\{L, L\}}$$

and

$$\frac{|\mathrm{Hom}(Y[-1], X)_L|}{|\mathrm{Aut} Y|} \cdot \frac{\{Y[-1], X\}}{\{Y, Y\}} = \frac{|\mathrm{Hom}(X, L)_Y|}{|\mathrm{Aut} L|} \cdot \frac{\{X, L\}}{\{L, L\}}.$$

Let \mathcal{A} be a finitary abelian category and $X, Y, L \in \mathcal{A}$. Define

$$E(X, Y; L) = \{(f, g) \in \mathrm{Hom}(X, L) \times \mathrm{Hom}(L, Y) \mid$$

$$0 \rightarrow X \xrightarrow{f} L \xrightarrow{g} Y \rightarrow 0 \text{ is an exact sequence}\}.$$

The group $\mathrm{Aut} X \times \mathrm{Aut} Y$ acts freely on $E(X, Y; L)$ and the orbit of $(f, g) \in E(X, Y; L)$ is denoted by $(f, g)^\wedge := \{(af, gc^{-1}) \mid (a, c) \in \mathrm{Aut} X \times \mathrm{Aut} Y\}$. If the orbit space is denoted by $O(X, Y; L) = \{(f, g)^\wedge \mid (f, g) \in E(X, Y; L)\}$, then the Hall number $g_{XY}^L = |O(X, Y; L)|$. It is easy to see

$$g_{XY}^L = \frac{|\mathcal{M}(X, L)_Y|}{|\mathrm{Aut} X|} = \frac{|\mathcal{M}(L, Y)_X|}{|\mathrm{Aut} Y|},$$

where $\mathcal{M}(X, L)_Y$ is the subset of $\mathrm{Hom}(X, L)$ consisting of monomorphisms $f : X \hookrightarrow L$ whose cokernels $\mathrm{Coker}(f)$ are isomorphic to Y and $\mathcal{M}(L, Y)_X$ is dually defined.

The equality in Corollary 2.2 can be regarded as a generalization of the Riedtmann-Peng formula in abelian categories to homologically finite triangulated categories. Indeed, assume that $\mathcal{C} = \mathcal{D}^b(\mathcal{A})$ for a finitary abelian category \mathcal{A} and X, Y and $L \in \mathcal{A}$. Then one can obtain

$$\mathrm{Hom}(Y, X[1])_{L[1]} = \mathrm{Ext}^1(Y, X)_L, \quad \{Y, X[1]\} = |\mathrm{Hom}_{\mathcal{A}}(Y, X)|^{-1}.$$

where $\mathrm{Ext}^1(X, Y)_L$ is the set of equivalence class of extension of Y by X with the middle term isomorphic to L and

$$g_{XY}^L = \frac{|\mathrm{Hom}(L, Y)_{X[1]}|}{|\mathrm{Aut} Y|}, \quad \{X, X\} = \{L, L\} = \{L, Y\} = 0.$$

Under the assumption, Corollary 2.2 is reduced to the Riedtmann-Peng formula ([15][12]).

$$\frac{|\mathrm{Ext}^1(Y, X)_L|}{|\mathrm{Hom}_{\mathcal{A}}(Y, X)|} = g_{XY}^L \cdot |\mathrm{Aut} X| \cdot |\mathrm{Aut} Y| \cdot |\mathrm{Aut} L|^{-1}.$$

3. TWO SYMMETRIES

Consider the following commutative square diagram in \mathcal{C} , which is pushout and pullback in meantime,

$$\begin{array}{ccc} L' & \xrightarrow{f'} & M \\ \downarrow m' & & \downarrow m \\ X & \xrightarrow{f} & L \end{array}$$

Applying the Octahedral axiom, one obtain the following commutative diagram:

$$(3.1) \quad \begin{array}{ccccccc} Z & \xlongequal{\quad} & Z & & & & \\ \vdots \downarrow l' & & \downarrow l & & & & \\ L' & \xrightarrow{f'} & M & \xrightarrow{g'} & Y & \xrightarrow{h'} & L'[1] \\ \downarrow m' & & \downarrow m & & \parallel & & \downarrow m'[1] \\ X & \xrightarrow{f} & L & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \\ \vdots \downarrow n' & & \downarrow n & & & & \\ Z[1] & \xlongequal{\quad} & Z[1] & & & & \end{array}$$

with rows and columns being distinguished triangles and a distinguished triangle

$$(3.2) \quad L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \xrightarrow{\begin{pmatrix} m \\ f \end{pmatrix}} L \xrightarrow{\theta} L'[1].$$

The triangle induces two sets

$$\begin{aligned} \mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]} &:= \{(m, f) \in \mathrm{Hom}(M \oplus X, L) \mid \\ &\quad \mathrm{Cone}(f) \simeq Y, \mathrm{Cone}(m) \simeq Z[1] \text{ and } \mathrm{Cone}(m, f) \simeq L'[1]\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}(L', M \oplus X)_L^{Y, Z[1]} &:= \{(f', -m') \in \mathrm{Hom}(L', M \oplus X) \mid \\ &\quad \mathrm{Cone}(f') \simeq Y, \mathrm{Cone}(m') \simeq Z[1] \text{ and } \mathrm{Cone}(f', -m') \simeq L\}. \end{aligned}$$

The symmetry-I: The orbit spaces of $\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}$ and $\mathrm{Hom}(L', M \oplus X)_L^{Y, Z[1]}$ under the action of $\mathrm{Aut} L$ and $\mathrm{Aut} L'$ respectively coincide. More explicitly, the symmetry implies the identity:

$$\frac{|\mathrm{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\mathrm{Aut} L|} \frac{|\{M \oplus X, L\}|}{|\{L', L\}\{L, L'\}|} = \frac{|\mathrm{Hom}(L', M \oplus X)_L^{Y, Z[1]}|}{|\mathrm{Aut} L'|} \frac{|\{L', M \oplus X\}|}{|\{L', L\}\{L', L'\}|}.$$

Proof. The equality is a direct application of Proposition 2.1 to the triangle 3.2. \square

Roughly speaking, **The symmetry-I** compares

$$L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \quad \text{and} \quad M \oplus X \xrightarrow{\begin{pmatrix} m \\ f \end{pmatrix}} L$$

in the triangle 3.2.

The diagram 3.1 induces **a new symmetry** which compares

$$L' \xrightarrow{f'} M \xrightarrow{m} L \quad \text{and} \quad L' \xrightarrow{m'} X \xrightarrow{f} L.$$

Using the derived Riedtmann-Peng formula (Corollary 2.2), we have

$$\begin{aligned} \frac{|\mathrm{Hom}(Y[-1], L')_M|}{|\mathrm{Aut} Y|} \cdot \frac{\{Y[-1], L'\}}{\{Y, Y\}} &= \frac{|\mathrm{Hom}(L', M)_Y|}{|\mathrm{Aut} M|} \cdot \frac{\{L', M\}}{\{M, M\}}, \\ \frac{|\mathrm{Hom}(L, Z[1])_{M[1]}|}{|\mathrm{Aut} Z|} \cdot \frac{\{L, Z[1]\}}{\{Z, Z\}} &= \frac{|\mathrm{Hom}(M, L)_{Z[1]}|}{|\mathrm{Aut} M|} \cdot \frac{\{M, L\}}{\{M, M\}}, \\ \frac{|\mathrm{Hom}(X, Z[1])_{L'[1]}|}{|\mathrm{Aut} Z|} \cdot \frac{\{X, Z[1]\}}{\{Z, Z\}} &= \frac{|\mathrm{Hom}(L', X)_{Z[1]}|}{|\mathrm{Aut} L'|} \cdot \frac{\{L', X\}}{\{L', L'\}}, \end{aligned}$$

and

$$\frac{|\mathrm{Hom}(Y[-1], X)_L|}{|\mathrm{Aut} Y|} \cdot \frac{\{Y[-1], X\}}{\{Y, Y\}} = \frac{|\mathrm{Hom}(X, L)_Y|}{|\mathrm{Aut} L|} \cdot \frac{\{X, L\}}{\{L, L\}}.$$

Hence, one can convert to compare

$$Y \xrightarrow{h'} L'[1], L \xrightarrow{n} Z[1] \quad \text{and} \quad X \xrightarrow{n'} Z[1], Y \xrightarrow{h} X[1]$$

in the diagram 3.1. In order to describe the second symmetry, we need to introduce some notations. Fix X, Y, Z, M, L and L' , define

$$\mathcal{D}_{L, L'} = \{(m, f, h, n) \in \mathrm{Hom}(M, L) \times \mathrm{Hom}(X, L) \times \mathrm{Hom}(Y, X[1]) \times \mathrm{Hom}(L, Z[1]) \mid (m, f, h, n) \text{ induces a diagram with the form as Diagram 3.1}\}$$

and

$$\mathcal{D}_{L', L} = \{(f', m', h', n') \in \mathrm{Hom}(L', X) \times \mathrm{Hom}(L', M) \times \mathrm{Hom}(Y, L'[1]) \times \mathrm{Hom}(X, Z[1]) \mid (m', f', h', n') \text{ induces a diagram with the form as Diagram 3.1}\}.$$

Here, we say “ (m, f, h, n) induces a diagram with the form as Diagram 3.1” means that there exist morphisms $m', f', h', n', g, g', l, l'$ such that all morphisms constitute a diagram formed as Diagram 3.1. The crucial point is that the square

$$\begin{array}{ccc} L' & \xrightarrow{f'} & M \\ \downarrow m' & & \downarrow m \\ X & \xrightarrow{f} & L \end{array}$$

is both pushout and pullback and rows and columns in Diagram 3.1 are distinguished triangles. Note that the pair (f', m') is uniquely determined by (m, f, h, n) up to isomorphisms under requirement, so the above notation is well-defined.

There exist natural projections:

$$p : \mathcal{D}_{L, L'} \rightarrow \mathrm{Hom}(Y, X[1]) \times \mathrm{Hom}(L, Z[1]),$$

$$i_1 : \mathrm{Hom}(Y, X[1]) \times \mathrm{Hom}(L, Z[1]) \rightarrow \mathrm{Hom}(Y, X[1])$$

and

$$i_2 : \mathrm{Hom}(Y, X[1]) \times \mathrm{Hom}(L, Z[1]) \rightarrow \mathrm{Hom}(L, Z[1]).$$

The image of $i_1 \circ p$ is denoted by $\mathrm{Hom}(Y, X[1])_{L[1]}^{L'}$ and given $h \in \mathrm{Hom}(Y, X[1])_{L[1]}^{L'}$, define $\mathrm{Hom}(L, Z[1])_{M[1]}^{h, L'}$ to be $i_2 \circ p^{-1} \circ i_1^{-1}(h)$. It is clear that

$$\mathrm{Hom}(Y, X[1])_{L[1]} = \bigsqcup_{[L']} \mathrm{Hom}(Y, X[1])_{L[1]}^{L'}.$$

Similarly, there exists projections:

$$q : \mathcal{D}_{L',L} \rightarrow \text{Hom}(Y, L'[1]) \times \text{Hom}(X, Z[1]),$$

$$j_1 : \text{Hom}(Y, L'[1]) \times \text{Hom}(X, Z[1]) \rightarrow \text{Hom}(Y, L'[1])$$

and

$$j_2 : \text{Hom}(Y, L'[1]) \times \text{Hom}(X, Z[1]) \rightarrow \text{Hom}(X, Z[1]).$$

The image of $j_1 \circ q$ is denoted by $\text{Hom}(X, Z[1])_{L'[1]}^L$ and for any $n' \in \text{Hom}(X, Z[1])_{L'[1]}$, denote $j_2 \circ p^{-1} \circ j_1^{-1}(n')$ by $\text{Hom}(Y, L'[1])_{M[1]}^{n',L}$.

The symmetry-II:

- Fix $h \in \text{Hom}(Y, X[1])_{L[1]}^{L'}$, then there exists a surjective map $f_* : \text{Hom}(L, Z[1])_{M[1]}^{h,L'} \rightarrow \text{Hom}(X, Z[1])_{L'[1]}^L$ satisfying that the cardinality of any fibre is

$$|(f_*)^{-1}| := |\text{Hom}(Y, Z[1])| \cdot \{X \oplus Y, Z[1]\} \cdot \{L, Z[1]\}^{-1};$$

- Fix $n' \in \text{Hom}(X, Z[1])_{L'[1]}^L$, then there exists a surjective map $(m')_* : \text{Hom}(Y, L'[1])_{M[1]}^{n',L} \rightarrow \text{Hom}(Y, X[1])_{L[1]}^{L'}$ satisfying that the cardinality of any fibre is

$$|(m')^{-1}| := |\text{Hom}(Y, Z[1])| \cdot \{Y, X[1] \oplus Z[1]\} \cdot \{Y, L'[1]\}^{-1};$$

- $|(f_*)^{-1}| \cdot \{Y, X[1]\} \cdot \{L, Z[1]\} = |(m')_*^{-1}| \cdot \{X, Z[1]\} \cdot \{Y, L'[1]\}.$

Proof. Given $h \in \text{Hom}(Y, X[1])_{L[1]}^{L'}$, there exists a triangle

$$\alpha : X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1].$$

Consider the action of the functor $\text{Hom}(-, Z[1])$ on the triangle, we obtain a long exact sequence

$$\cdots \longrightarrow \text{Hom}(Y, Z[1]) \xrightarrow{u} \text{Hom}(L, Z[1]) \xrightarrow{v} \text{Hom}(X, Z[1]) \longrightarrow \cdots$$

Then the cardinality of $\text{Im}(u)$ is $|\text{Hom}(Y, Z[1])| \cdot \{X \oplus Y, Z[1]\} \cdot \{L, Z[1]\}^{-1}$. The map f_* is the restriction of v to $\text{Hom}(L, Z[1])_{M[1]}^{h,L'}$. By definition, f_* is epic and the fibre is isomorphic to $\text{Ker}(v) = \text{Im}(u)$. Thus the first statement obtained. The second statement is proved in the same way. The third statement is a direct confirmation. \square

Note that there also exists projections:

$$p_{12} : \mathcal{D}_{L,L'} \rightarrow \text{Hom}(M \oplus X, L)$$

with the image $\text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}$ and

$$q_{12} : \mathcal{D}_{L',L} \rightarrow \text{Hom}(L', M \oplus X)$$

with the image $\text{Hom}(L', M \oplus X)_L^{Y,Z[1]}$. **The symmetry-I** characterizes the relation between Imp_{12} and $\text{Im}q_{12}$. Meanwhile, **The symmetry-II** characterizes the relation between Imp and $\text{Im}q$. The relation between Imp_{12} and Imp ($\text{Im}q_{12}$ and $\text{Im}q$) is implicitly shown by the derived Riedtmann-Peng formula (Corollary 2.2). More explicitly, consider the projections

$$t_1 : \text{Hom}(M \oplus X, L) \rightarrow \text{Hom}(X, L) \text{ and } t_2 : \text{Hom}(M \oplus X, L) \rightarrow \text{Hom}(M, L)$$

and

$$s_1 : \text{Hom}(L', M \oplus X) \rightarrow \text{Hom}(L', X) \text{ and } s_2 : \text{Hom}(L', M \oplus X) \rightarrow \text{Hom}(L', M).$$

Using Corollary 2.2, we obtain

$$\frac{|\text{Hom}(Y, X[1])_{L[1]}^{L'}|}{|\text{Aut}Y|} \cdot \frac{\{Y, X[1]\}}{\{Y, Y\}} = \frac{|\text{Im}t_1 \circ p_{12}|}{|\text{Aut}L|} \cdot \frac{\{X, L\}}{\{L, L\}}.$$

Given a triangle $\alpha : X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1]$ with $h \in \text{Hom}(Y, X[1])_{L[1]}^{L'}$, applying Corollary 2.2 again, we have

$$\frac{|\text{Hom}(L, Z[1])_{M[1]}^{h, L'}|}{|\text{Aut}Z|} \cdot \frac{\{L, Z[1]\}}{\{Z, Z\}} = \frac{|t_2 \circ p_{12}^{-1} \circ t_1^{-1}(f)|}{|\text{Aut}M|} \cdot \frac{\{M, L\}}{\{M, M\}}.$$

In the same way, we have

$$\frac{|\text{Hom}(X, Z[1])_{L'[1]}^L|}{|\text{Aut}Z|} \cdot \frac{\{X, Z[1]\}}{\{Z, Z\}} = \frac{|\text{Im}s_1 \circ q_{12}|}{|\text{Aut}L'|} \cdot \frac{\{L', X\}}{\{L', L'\}},$$

and

$$\frac{|\text{Hom}(Y, L'[1])_{M[1]}^{n', L}|}{|\text{Aut}Y|} \cdot \frac{\{Y, L'[1]\}}{\{Y, Y\}} = \frac{|s_2 \circ q_{12}^{-1} \circ s_1^{-1}(f)|}{|\text{Aut}M|} \cdot \frac{\{L', M\}}{\{M, M\}}.$$

The above four identities induces the equivalence of **The symmetry-I** and **The symmetry-II**.

4. TWO VERSIONS OF DERIVED HALL ALGEBRAS

For any X, Y and $L \in \mathcal{C}$, set

$$F_{XY}^L = \frac{|\text{Hom}(L, Y)_{X[1]}|}{|\text{Aut}Y|} \cdot \frac{\{L, Y\}}{\{Y, Y\}} = \frac{|\text{Hom}(X, L)_Y|}{|\text{Aut}X|} \cdot \frac{\{X, L\}}{\{X, X\}}.$$

Theorem 4.1. ([19],[20]) *Let $\mathcal{H}(\mathcal{C})$ be the vector space over \mathbb{Q} with the basis $\{u_{[X]} \mid X \in \mathcal{C}\}$. Endowed with the multiplication defined by*

$$u_{[X]} * u_{[Y]} = \sum_{[L]} F_{XY}^L u_{[L]},$$

$\mathcal{H}(\mathcal{C})$ is an associative algebra with the unit $u_{[0]}$.

We recall the proof sketch of the theorem. To prove $u_{[Z]} * (u_{[X]} * u_{[Y]}) = (u_{[Z]} * u_{[X]}) * u_{[Y]}$ is equivalent to prove

$$\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L']} F_{ZX}^{L'} F_{L'Y}^M.$$

It is direct to check

$$\text{LHS} = \frac{1}{|\text{Aut}X| \cdot \{X, X\}} \sum_{[L]} \sum_{[L']} \frac{|\text{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\text{Aut}L|} \cdot \frac{\{M \oplus X, L\}}{\{L, L\}}.$$

$$\text{RHS} = \frac{1}{|\text{Aut}X| \cdot \{X, X\}} \sum_{[L']} \sum_{[L]} \frac{|\text{Hom}(L', M \oplus X)_L^{Y, Z[1]}|}{|\text{Aut}L'|} \cdot \frac{\{L', M \oplus X\}}{\{L', L'\}}.$$

The symmetry-I naturally deduces LHS=RHS and then complete the proof of Theorem 4.1.

In [9], Kontsevich and Soibelman defined the motivic Hall algebra for an ind-constructible triangulated A_∞ -category. It is easy to extend the definition to a homologically finite triangulated category over k . Define $\mathcal{H}_{\mathcal{M}}(\mathcal{C})$ to be the vector space over \mathbb{Q} with the basis $\{v_{[X]} \mid X \in \mathcal{C}\}$, endowed with the multiplication defined by

$$\begin{aligned} v_{[X]} * v_{[Y]} &= \{Y, X[1]\} \cdot \sum_{[L]} |\mathrm{Hom}(Y, X[1])_{L[1]}| v_{[L]} \\ &= \{Y[-1], X\} \cdot \sum_{[L]} |\mathrm{Hom}(Y[-1], X)_L| v_{[L]} \end{aligned}$$

for any X, Y and $L \in \mathcal{C}$.

The symmetry-II naturally deduces the associativity of $\mathcal{H}_{\mathcal{M}}(\mathcal{C})$.

Theorem 4.2. *The algebra $\mathcal{H}_{\mathcal{M}}(\mathcal{C})$ is associative.*

Proof. For any X, Y and Z in \mathcal{C} , we need to prove

$$v_{[Z]} * (v_{[X]} * v_{[Y]}) = (v_{[Z]} * v_{[X]}) * v_{[Y]}.$$

By the definition of the multiplication, it is equivalent to prove

$$\begin{aligned} &\sum_{[L]} \{Y, X[1]\} \{L, Z[1]\} |\mathrm{Hom}(Y, X[1])_{L[1]}| \cdot |\mathrm{Hom}(L, Z[1])_{M[1]}| \\ &= \sum_{[L']} \{X, Z[1]\} \{Y, L'[1]\} |\mathrm{Hom}(X, Z[1])_{L'[1]}| \cdot |\mathrm{Hom}(Y, L'[1])_{M[1]}|. \end{aligned}$$

Following the first statement of **The symmetry-II**, LHS is equal to

$$\begin{aligned} &\sum_{[L], [L']} \{Y, X[1]\} \{L, Z[1]\} |\mathrm{Hom}(Y, X[1])_{L[1]}^{L'}| \cdot |\mathrm{Hom}(X, Z[1])_{L'[1]}^L| \cdot \\ &\quad |\mathrm{Hom}(Y, Z[1])| \cdot \{X \oplus Y, Z[1]\} \cdot \{L, Z[1]\}^{-1}. \end{aligned}$$

Following the second statement of **The symmetry-II**, RHS is equal to

$$\begin{aligned} &\sum_{[L'], [L]} \{X, Z[1]\} \{Y, L'[1]\} |\mathrm{Hom}(X, Z[1])_{L'[1]}^L| \cdot |\mathrm{Hom}(Y, X[1])_{L[1]}^{L'}| \cdot \\ &\quad |\mathrm{Hom}(Y, Z[1])| \cdot \{Y, X[1] \oplus Z[1]\} \cdot \{Y, L'[1]\}^{-1}. \end{aligned}$$

The equality LHS=RHS is just the third statement of **The symmetry-II**. \square

Theorem 4.3. *The map $\Phi : \mathcal{H}_{\mathcal{M}}(\mathcal{C}) \rightarrow \mathcal{H}(\mathcal{C})$ by $\Phi(v_{[X]}) = |\mathrm{Aut} X| \cdot \{X, X\} \cdot u_{[X]}$ for any $X \in \mathcal{C}$ is an algebraic isomorphism between $\mathcal{H}_{\mathcal{M}}(\mathcal{C})$ and $\mathcal{H}(\mathcal{C})$.*

Proof. It is obvious that Φ is a bijection. In order to prove the theorem, it is enough to show that Φ is an algebraic morphism. For any $X, Y \in \mathcal{C}$, we have $\Phi(v_{[X]}) * \Phi(v_{[Y]})$ is equal to

$$\begin{aligned} &(|\mathrm{Aut} X| \cdot \{X, X\} \cdot u_{[X]}) \cdot (|\mathrm{Aut} Y| \cdot \{Y, Y\} \cdot u_{[Y]}) \\ &= \sum_{[L]} \frac{|\mathrm{Hom}(L, Y)_{X[1]}|}{|\mathrm{Aut} L|} \cdot |\mathrm{Aut} X| \frac{\{L, Y\} \cdot \{X, X\}}{\{L, L\}} \cdot (|\mathrm{Aut} L| \cdot \{L, L\} \cdot u_{[L]}) \\ &= \{Y, X[1]\} \cdot \sum_{[L]} |\mathrm{Hom}(Y, X[1])_{L[1]}| \cdot (|\mathrm{Aut} L| \cdot \{L, L\} \cdot u_{[L]}) \\ &= \Phi(v_{[X]} * v_{[Y]}). \end{aligned}$$

Here, the second equality follows from Corollary 2.2. \square

5. MOTIVIC HALL ALGEBRAS

The aim of this section is to give an alternative proof of [9, Proposition 10] suggested by the proof of Theorem 4.2. We recall the definition of motivic Hall algebras in [9]. In the following, fix the complex field \mathbb{C} . A ind-constructible set is a countable union of non-intersecting constructible sets.

Example 5.1. [2, 21] Let \mathbb{C} be the complex field and A be a finite dimensional algebra $\mathbb{C}Q/I$ with indecomposable projective modules P_i , $i = 1, \dots, l$. Given a projective complex $P^\bullet = (P^i, \partial_i)_{i \in \mathbb{Z}}$ with $P^i = \bigoplus_{j=1}^l e_j^i P_j$. We denote by \underline{e} the vector $(e_1^i, e_2^i, \dots, e_l^i)$. The sequence, denoted by $\underline{\mathbf{e}} = \underline{\mathbf{e}}(P^\bullet) = (\underline{e}^i)_{i \in \mathbb{Z}}$ is called the projective dimension sequence of P^\bullet . We assume that only finitely many \underline{e}^i in $\underline{\mathbf{e}}$ is nonzero. Define $\mathcal{P}(A, \underline{\mathbf{e}})$ to be the subset of

$$\prod_{i \in \mathbb{Z}} \text{hom}_A(P^i, P^{i+1}) = \prod_{i \in \mathbb{Z}} \text{hom}_A\left(\bigoplus_{j=1}^l e_j^i P_j, \bigoplus_{j=1}^l e_j^{i+1} P_j\right)$$

which consists of elements $(\partial_i : P^i \rightarrow P^{i+1})_{i \in \mathbb{Z}}$ such that $\partial_{i+1} \partial_i = 0$ for all $i \in \mathbb{Z}$. It is an affine variety with a natural action of the algebraic group $G_{\underline{\mathbf{e}}} = \prod_{i \in \mathbb{Z}} \text{Aut}_A(P^i)$. Let $K_0(\mathcal{D}^b(A))$, or simply by K_0 , be the Grothendieck group of the derived category $\mathcal{D}^b(A)$, and $\underline{\dim} : \mathcal{D}^b(A) \rightarrow K_0(\mathcal{D}^b(A))$ the canonical surjection. It induces a canonical surjection from the abelian group of dimension vector sequences to K_0 , we still denote it by $\underline{\dim}$. Given $\mathbf{d} \in K_0$, the set

$$\mathcal{P}(A, \mathbf{d}) = \bigsqcup_{\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{P}(A, \underline{\mathbf{e}})$$

is an ind-constructible set.

Let \mathcal{X} be a constructible stack over \mathbb{C} . The set $\text{Mot}(\mathcal{X})$ is an abelian group generated by isomorphism classes $[\pi : \mathcal{S} \rightarrow \mathcal{X}]$ of morphisms to \mathcal{X} satisfying:

- $[(\mathcal{S}_1 \sqcup \mathcal{S}_2) \rightarrow \mathcal{X}] = [\mathcal{S}_1 \rightarrow \mathcal{X}] + [\mathcal{S}_2 \rightarrow \mathcal{X}]$
- $[\pi_1 : \mathcal{S}_1 \rightarrow \mathcal{X}] = [\pi_2 : \mathcal{S}_2 \rightarrow \mathcal{X}]$ if \exists Zariski fibrations $f_i : \mathcal{S}_i \rightarrow \mathcal{S}$, $i = 1, 2$ and $h : \mathcal{S} \rightarrow \mathcal{X}$ with $\pi_i = h \circ f_i$.

$\text{Mot}(\mathcal{X})$ is naturally the $\text{Mot}(\text{Spec}(\mathbb{C}))$ -module. Denote by \mathbb{L} the identity element in $\text{Mot}(\text{Spec}(\mathbb{C}))$ and $\text{Mot}(\mathcal{X})[\mathbb{L}^{-1}]$ the localization of $\text{Mot}(\mathcal{X})$.

In the following, we assume that

- objects in \mathcal{C} form an ind-constructible set $\mathfrak{Obj}(\mathcal{C}) = \bigsqcup_{i \in I} \mathcal{X}_i$ for countable constructible stacks \mathcal{X}_i with the action of an affine algebraic group G_i .
- For any $\mathcal{X}_i, \mathcal{X}_j$, $\exists n(i, j) \in \mathbb{N}$ such that $|\{x_i, x_j\}| \leq n(i, j)$ for $x_i \in \mathcal{X}_i, x_j \in \mathcal{X}_j$.

For example, if $\mathcal{A} = \text{mod} A$ for a finite dimensional algebra A over \mathbb{C} , the above assumptions hold for $\mathcal{D}^b(\mathcal{A})$. The quotient stack of \mathcal{X}_i by G_i is denoted by $[\mathcal{X}_i/G_i]$. Define

$$\mathcal{MH}(\mathcal{C}) = \bigoplus_{i \in I} \text{Mot}([\mathcal{X}_i/G_i][\mathbb{L}^{-1}])$$

with the multiplication

$$[\pi_1 : \mathcal{S}_1 \rightarrow \mathfrak{Obj}(\mathcal{C})] \cdot [\pi_2 : \mathcal{S}_2 \rightarrow \mathfrak{Obj}(\mathcal{C})] = \sum_{n \in \mathbb{Z}} [\pi : \mathcal{W}_n \rightarrow \mathfrak{Obj}(\mathcal{C})] \mathbb{L}^{-n}$$

where

$$\mathcal{W}_n = \{(s_1, s_2, \alpha) \mid s_i \in \mathcal{S}_i, \alpha \in \text{Hom}_{\mathcal{C}}(\pi_2(s_2), \pi_1(s_1)[1]), \\ \sum_{i>0} (-1)^i \dim_{\mathbb{C}} \text{Hom}(\pi_2(s_2)[i], \pi_1(s_1)[1]) = n.\}$$

The map π sends (s_1, s_2, α) to $\text{Cone}(\alpha)[-1]$. For $X, Y \in \mathfrak{D}\mathbf{b}\mathbf{j}(\mathcal{C})$, set

$$\{X, Y\} := \mathbb{L} \sum_{i>0} (-1)^i \dim_{\mathbb{C}} \text{Hom}(X[i], Y).$$

For convenience, we use the integral notation for the right-hand term in the definition of the multiplication. Then the multiplication can be rewritten as

$$\begin{aligned} & [\pi_1 : \mathcal{S}_1 \rightarrow \mathfrak{D}\mathbf{b}\mathbf{j}(\mathcal{C})] \cdot [\pi_2 : \mathcal{S}_2 \rightarrow \mathfrak{D}\mathbf{b}\mathbf{j}(\mathcal{C})] \\ &:= \int_{s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_1} [\text{Hom}_{\mathcal{C}}(\pi_2(s_2), \pi_1(s_1)[1]) \rightarrow \mathfrak{D}\mathbf{b}\mathbf{j}(\mathcal{C})] \cdot \{\pi_2(s_2), \pi_1(s_1)[1]\} \\ &:= \int_{s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_1} \{\pi_2(s_2), \pi_1(s_1)[1]\} \cdot \int_{\alpha \in \text{Hom}_{\mathcal{C}}(\pi_2(s_2), \pi_1(s_1)[1])_E} v_{[E]} \end{aligned}$$

where $v_{[E]} := [\pi : pt \rightarrow \mathfrak{D}\mathbf{b}\mathbf{j}(\mathcal{C})]$ with $\pi(pt) = E$.

Theorem 5.2. [9, Proposition 10] *With the above multiplication, $\mathcal{MH}(\mathcal{C})$ becomes an associative algebra.*

Inspired by [9] and [20], the proof is a motivic version of **The symmetry-II**.

Proof. By the reformulation of the multiplication definition, the proof of the theorem is easily reduced to the case that \mathcal{S}_i is just a point. Given X, Y and $Z \in \mathfrak{D}\mathbf{b}\mathbf{j}(\mathcal{C})$, $v_{[Z]} * (v_{[X]} * v_{[Y]})$ is equal to

$$\mathcal{T}_1 := \int_{\alpha \in \text{Hom}(Y, X[1])_{L[1]}} \int_{\beta \in \text{Hom}(L, Z[1])_{M[1]}} \{Y, X[1]\} \cdot \{L, Z[1]\} \cdot v_{[M]}$$

and $(v_{[Z]} * v_{[X]}) * v_{[Y]}$ is equal to

$$\mathcal{T}_2 := \int_{\alpha' \in \text{Hom}(X, Z[1])_{L'[1]}} \int_{\beta' \in \text{Hom}(Y, L'[1])_{M[1]}} \{X, Z[1]\} \cdot \{Y, L'[1]\} \cdot v_{[M]}.$$

Using the notation in Section 3, we have

$$\mathcal{T}_1 = \int_{\alpha' \in \text{Hom}(X, Z[1])_{L'[1]}^L, \alpha \in \text{Hom}(Y, X[1])_{L[1]}^{L'}} \int_{\beta \in \text{Hom}(L, Z[1])_{M[1]}^{\alpha, L'}} \{Y, X[1]\} \cdot \{L, Z[1]\} \cdot v_{[M]}$$

and

$$\mathcal{T}_2 = \int_{\alpha \in \text{Hom}(Y, X[1])_{L[1]}^{L'}, \alpha' \in \text{Hom}(X, Z[1])_{L'[1]}^L} \int_{\beta' \in \text{Hom}(Y, L'[1])_{M[1]}^{\alpha', L}} \{X, Z[1]\} \cdot \{Y, L'[1]\} \cdot v_{[M]}.$$

As the proof of Theorem 4.2, fix $\alpha \in \text{Hom}(Y, X[1])_{L[1]}^{L'}$, by the diagram 3.1, there is a constructible bundle $\text{Hom}(L, Z[1])_{M[1]}^{\alpha, L'} \rightarrow \text{Hom}(X, Z[1])_{L'[1]}^L$ with fibre dimension

$$\dim \text{Hom}(Y, Z[1]) + \dim \{X \oplus Y, Z[1]\} - \dim \{L, Z[1]\}.$$

Fix $\alpha' \in \text{Hom}(X, Z[1])_{L'[1]}^L$, by the diagram 3.1, there is a constructible bundle $\text{Hom}(Y, L'[1])_{M[1]}^{\alpha', L} \rightarrow \text{Hom}(Y, X[1])_{L[1]}^{L'}$ with fibre dimension

$$\dim \text{Hom}(Y, Z[1]) + \dim \{Y, X[1] \oplus Z[1]\} - \dim \{Y, L'[1]\}.$$

Hence, we have $\mathcal{T}_1 = \mathcal{T}_2$. □

REFERENCES

- [1] T. Bridgeland, *An introduction to motivic Hall algebras*, arXiv:1002.4372.
- [2] B. Jensen, X. Su and A. Zimmermann, *Degeneration for derived categories*, J. Pure and Applied Algebra, **198**(2005), 281–295.
- [3] D. Joyce, *Configurations in abelian categories. I. Basic properties and moduli stacks*, Adv. Math. **203** (2006), 194–255.
- [4] D. Joyce, *Configurations in abelian categories. II. Ringel-Hall algebras*, Adv. Math. **210** (2007), 635–706.
- [5] D. Joyce, *Configurations in abelian categories. III. Stability conditions and identities*, Adv. Math. **215** (2007), 153–219.
- [6] D. Joyce, *Configurations in abelian categories. IV. Invariants and changing stability conditions*, Adv. Math. **217** (2008), 125–204.
- [7] D. Joyce and Y. Song, *A theory of generalized Donaldson-Thomas invariants*, arXiv:0810.5645.
- [8] M. Kapranov, *Heisenberg doubles and derived categories*, J. Algebra **202** (1998), no. 2, 712–744.
- [9] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.
- [10] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.
- [11] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), 515–560.
- [12] L. Peng, *Lie algebras determined by finite Auslander-Reiten quivers*, Comm. in Alg. **26** (1998), no.9, 2711–2725.
- [13] L. Peng and J. Xiao, *Root categories and simple Lie algebras*, J. Algebra. **198** (1997), 19–56.
- [14] L. Peng and J. Xiao, *Triangulated categories and Kac-Moody algebras*, Invent. Math. **140** (2000), 563–603.
- [15] Ch. Riedtmann, *Lie algebras generated by indecomposables*, J. Algebra **170** (1994), no. 2, 526–546.
- [16] M. Reineke, *The Harder-Narasimhan system in quantum groups and cohomology of quiver moduli*, Invent. Math. **152** (2003), 349–368.
- [17] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–592.
- [18] E. Steinitz, *Zur Theorie der Abel'schen Gruppen*, Jahresberichts der DMV **9** (1901), 80–85.
- [19] B. Toën, *Derived Hall algebras*, Duke Math. J. **135** (2006), no. 3, 587–615.
- [20] J. Xiao and F. Xu, *Hall algebras associated to triangulated categories*, Duke Math. J. **143** (2008), no. 2, 357–373.
- [21] J. Xiao, F. Xu and G. Zhang, *Derived categories and Lie algebras*, arXiv:0604564.

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